CALCULATION OF SUPERCAVITATIONAL FLOW PAST SLENDER PROFILES IN PROXIMITY TO THE SEPARATION BOUNDARY

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Numerical data on pressure distribution, cavity thickness, and the overall characteristics of fully cavitated slender profiles moving near the free surface of a weightless liquid or close to a solid wall are obtained on the basis of the potential of accelerations and of approximate solution of singular integral equations by the method of discrete perturbations.

The problem of steady motion of a fully cavitating profile near a free surface was considered in linear formulation by Johnson [1], Auslaender [2], and Yim [3]. The method of conformal mapping was used in their papers for deriving the expansion of the complex potential in the neighborhood of an infinitely distant point. Analytical expressions for the over-all characteristics were also obtained for the case of an unbounded cavity [1, 2].

The mathematics of this problem reduce to the determination of the velocity potential with discontinuities of tangent and normal derivatives, or of that of accelerations with discontinuities of the function itself and of normal derivatives. The analysis of both problems leads to the same conclusions.

1. Let us consider the following pattern of cavitational flow past a slim profile.

Let the cavity begin over the leading edge, run off from below the trailing edge, and be closed in an elliptical contour at a certain distance $l > 1$ from the leading edge.

We introduce an orthogonal system of coordinates rigidly attached to the body with its axis of abscissas directed along the flow velocity v_∞ of the unperturbed stream and the axis of ordinates vertically upward.

For the potential of accelerations or pressure the boundary value problem is formulated thus:

$$\Delta P = 0, \qquad P = \frac{P - P_\infty}{{}^1\!/_2 \rho v_\infty^2} \tag{1.1}$$

outside the profile and the cavity

The kinematic condition along the wetted part of the profile and at the cavity boundary yields

$$2 \frac{d}{dx}\left[f_c(x) \pm \frac{1}{2} t(x)\right] = -\lim_{y \to 0\pm} \int_{-\infty}^{x} \frac{\partial P}{\partial y}(\lambda, y)\, d\lambda \tag{1.2}$$

Here $f_c(x)$ is the mean line of the profile and of the cavern, and $t(x)$ is the thickness of the cavern.

At the upper and lower boundaries of the cavern the dynamic condition of pressure constancy

$$P\,|_{y \to 0^+} \to -\varkappa, \qquad \varkappa = \frac{P_\infty - P_k}{{}^1\!/_2 \rho v_\infty^2} \tag{1.3}$$

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where $\varkappa$ is the cavitation number, and P_∞ and P_k are pressures at infinity and in the cavity respectively, must also be satisfied.

These boundary conditions must be supplemented by the condition of absence of perturbations at infinity and the condition at the free surface ($y = h$) or at the solid wall ($y = -h$).

These are, respectively,

$$P|_{y=h} = 0, \qquad \frac{\partial P}{dy}\Big|_{y=-h} = 0 \tag{1.4}$$

Using the Green's formulas and the method of conformal mapping with respect to either a free surface or a solid wall, we represent the solution of the stated boundary value problem (1.1)-(1.4) for the potential of accelerations or pressure in the form

$$P(x, y) = -\frac{1}{2\pi}\int_0^l (\ln r + \nu \ln r_1)\left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi - \frac{1}{2\pi}\int_0^1 \frac{\partial}{\partial y}[\ln r - \nu \ln r_1]\,\gamma(\xi)\, d\xi \tag{1.5}$$

$$r = \sqrt{(x-\xi)^2 + y^2}, \qquad r_1 = \sqrt{(x-\xi)^2 + (y + 2\nu h)^2}$$
$$\gamma(x) = P_-(x) - P_+(x), \qquad \left[\frac{\partial P}{\partial \eta}\right] = \frac{\partial P}{\partial \eta}\Big|_- - \frac{\partial P}{\partial \eta}\Big|_+ \tag{1.6}$$

Here $\nu = -1$ in the case of free surface and $\nu = +1$ for a solid wall; $\gamma(x)$ is the jump of pressure P, and $|dP/d\eta|$ is the jump of the normal derivative of the pressure.

The kernels of integrals in formula (1.5) have been constructed with the boundary conditions at the free surface and at the wall taken into account.

With boundary conditions (1.2) and (1.4) fulfilled we obtain the system of integral equations

$$2\frac{d}{dx}\left[f_c(x) + \frac{1}{2}t(x)\right] = \frac{1}{2}\int_0^x \left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi$$
$$+ \frac{1}{2\pi}\int_0^l \operatorname{arc\,tg}\frac{x-\xi}{2h}\left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi + \frac{1}{4}\int_0^l \left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi$$
$$- \frac{1}{2\pi}\int_0^1 \left[\frac{1}{x-\xi} - \nu\frac{x-\xi}{(x-\xi)^2 + 4h^2}\right]\gamma(\xi)\, d\xi$$
$$2\frac{d}{dx}\left[f_c(x) - \frac{1}{2}t(x)\right] = -\frac{1}{2}\int_0^x \left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi + \frac{1}{2\pi}\int_0^l \operatorname{arc\,tg}\frac{x-\xi}{2h}\left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi$$
$$+ \frac{1}{4}\int_0^l \left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi - \frac{1}{2\pi}\int_0^1 \left[\frac{1}{x-\xi} - \nu\frac{x-\xi}{(x-\xi)^2 + 4h^2}\right]\gamma(\xi)\, d\xi$$
$$-\varkappa = -\frac{1}{2\pi}\int_0^l \left(\ln|x-\xi| + \nu \ln\sqrt{(x-\xi)^2 + 4h^2}\right)\left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi$$
$$-\frac{1}{2}\gamma(x) + \frac{1}{2\pi}\int_0^1 \frac{2h}{(x-\xi)^2 + 4h^2}\gamma(\xi)\, d\xi \tag{1.7}$$

Subtracting the second equation from the first, we obtain

$$2t'(x) = \int_0^x \left[\frac{\partial P}{\partial \eta}\right](\xi)\, d\xi = 2q(x) \tag{1.8}$$

Stipulating the condition $q(0) = q(l) = 0$ and taking (1.8) into consideration, we integrate the second and third equation of system (1.7):

$$\frac{d}{dx}\left[f_c(x) - \frac{1}{2}t(x)\right] = -\frac{1}{2}q(x) + \frac{1}{2\pi}\int_0^l \frac{2h}{(x-\xi)^2 + 4h^2} q(\xi)\, d\xi - \frac{1}{4\pi}\int_0^1 \gamma(\xi)\left[\frac{1}{x-\xi} - \nu\frac{x-\xi}{(x-\xi)^2 + 4h^2}\right] d\xi$$

$$\varkappa = \frac{1}{\pi}\int_0^l q(\xi)\left[\frac{1}{x-\xi} + \nu\frac{x-\xi}{(x-\xi)^2+4h^2}\right]d\xi + \frac{1}{2}\gamma(x) - \frac{1}{2\pi}\int_0^1 \frac{2h\gamma(\xi)\,d\xi}{(x-\xi)^2+4h^2}$$

$$\int_0^l q(\xi)\,d\xi = t(l) = 0 \qquad (1.9)$$

The last relationship is the condition for point closing of the cavity, which at that point has a vertical tangent.

The equations of system (1.9) are particular integral equations with Cauchy kernels of the general kind. The unknown densities of integrals $\gamma(x)$ and $q(x)$ of system (1.9) may be considered to be intensities of the double and single layers distributed along segments $|0, 1|$ and $|0, l|$, respectively. It follows from physical considerations that the solution of system (1.9) is to be sought in the class of functions such that: function γ has an integrable singularity at $x = 0$ and is bounded for $x = 1$ (Zhukovskii-Chaplygin condition); function q is bounded for $x = 0$ and has an integrable singularity for $x = l$.

An analytical solution of system (1.9) has been found only for the case of transition to limit ($h \to \infty$ and $l \to \infty$) [4]. One of the numerical solutions of system (1.9) and the derivation of its basis of hydrodynamic characteristics are considered below.

For solving system (1.9) we use the method of discrete perturbations, which is a generalization of the 3/4 method first used by Pistolezzi, Weissinger, and Folkner and fully developed by Belotserkovskii and his students [5] under the name of discrete vortices. The method consists of the substitution of a discrete distribution of perturbations in layers for the continuous one and the derivation on this basis of a system of linear algebraic equations by using mechanical squaring formulas. The distribution order of perturbations and point s at which boundary conditions are satisfied is determined by the class of functions in which the solution is being sought.

2. As shown in [4, 6, 7] the pressure drop near the leading edge of a cavitating profile has for $\delta \to 0$ a singularity of order $\delta^{-1/4}$. The method of discrete vortices provides a good approximate solution of the problem of a noncavitating profile, if for $\delta \to 0$ the solution singularity at the leading edge is of the order

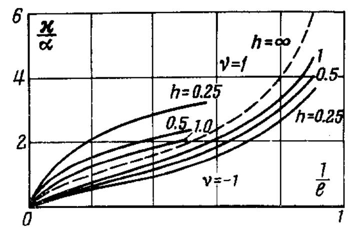

Fig. 1

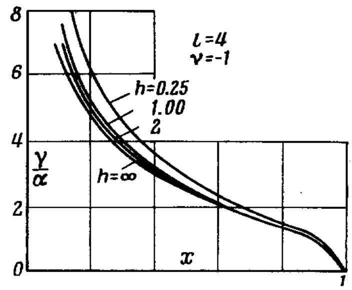

Fig. 2

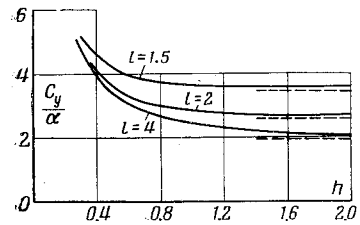

Fig. 3

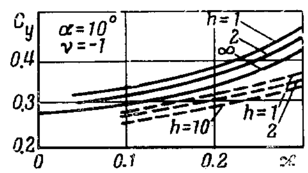

Fig. 4

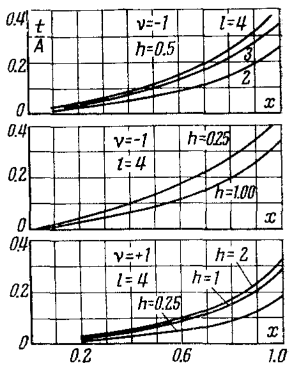

Fig. 5

of $\delta^{-1/2}$. Therefore to improve the convergence of the 3/4 method we first pass in (1.9) to variables $\xi = t^2$ and $x = z^2$

$$\frac{d}{dx}\left[f_c(x) - \frac{1}{2}t(x)\right]\Big|_{x=z^2} = -\frac{1}{2}q(z)$$

$$+\frac{1}{\pi}\int_0^{\sqrt{l}} \frac{2ht}{(z^2-t^2)^2+4h^2} q(t)\,dt - \frac{1}{2\pi}\int_0^1 \gamma(t)\left[\frac{1}{z^2-t^2} - \nu\frac{z^2-t^2}{(z^2-t^2)^2+4h^2}\right]t\,dt$$

$$\varkappa = \frac{2}{\pi}\int_0^{\sqrt{l}} q(t)\left[\frac{1}{z^2-t^2} + \nu\frac{z^2-t^2}{(z^2-t^2)^2+4h^2}\right]t\,dt + \frac{1}{2}\gamma(z) - \frac{1}{\pi}\int_0^1 \frac{2ht}{(z^2-t^2)^2+4h^2}\gamma(t)\,dt$$

$$\int_0^{\sqrt{l}} q(t)\,t\,dt = 0 \qquad (2.1)$$

If N is the number of segments into which the chord length of profile $|0, 1|$ is subdivided and M is the number of segments of line $|1, \sqrt{l}\,|$, the selection of the singularity and calculation points is made as follows:

for $\gamma(t)$

$$z_i = \left(i - \frac{1}{4}\right)\frac{1}{N}, \quad t_j = \left(j - \frac{3}{4}\right)\frac{1}{N} \quad (i, j = 1, 2, \ldots, N)$$

for $q(t)$

$$z_k = \left(k - \frac{3}{4}\right)\frac{1}{N}, \quad t_m = \left(m - \frac{1}{4}\right)\frac{1}{N} \quad (k, m = 1, 2, \ldots, N)$$

$$z_k = 1 + \left(k - \frac{3}{4}\right)\frac{\sqrt{l}-1}{M}, \quad t_m = 1 + \left(m - \frac{1}{4}\right)\frac{\sqrt{l}-1}{M} \quad (k, m = 1, 2, \ldots, M) \qquad (2.2)$$

Convergence of the described method is investigated by comparing the exact and the approximate solutions obtained for various N and M in the case of an unbounded liquid.

In the hydrodynamics of cavitational flows the following two problems are of practical interest: 1) determination of the shape of the cavity and of hydrodynamic characteristics for a given body shape; 2) determination of the shape of the body and of the cavity, and of the hydrodynamic characteristics for a given pressure (load) distribution on the body.

Let us consider two examples.

Example 1. A plate at an angle of attack α. In this case

$$\frac{d}{dx}\left[f_c(x) - \frac{1}{2}t(x)\right] = -\alpha$$

We substitute the system of linear algebraic equations

$$\frac{1}{N}\sum_{j=1}^{N}\gamma_j\left[\frac{1}{z_i^2-t_j^2} - \nu\frac{z_i^2-t_j^2}{(z_i^2-t_j^2)^2+4h^2}\right]t_j + \pi q_i - \frac{1}{N}\sum_{m=1}^{N}\frac{4ht_m q_m}{(z_i^2-t_m^2)^2+4h^2}$$

$$-\frac{\sqrt{l}-1}{M}\sum_{m=1}^{M}\frac{4ht_m q_{N+m}}{(z_i^2-t_m^2)^2+4h^2} = 2\pi\varkappa \quad (i = 1, 2, \ldots, N)$$

$$\frac{\pi}{2}\gamma_k - \frac{1}{N}\sum_{j=1}^{N}\frac{2ht_j\gamma_j}{(z_k^2-t_j^2)^2+4h^2} + \frac{2}{N}\sum_{m=1}^{N}q_m\left[\frac{1}{z_k^2-t_m^2} + \frac{\nu(z_k^2-t_m^2)}{(z_k^2-t_m^2)^2+4h^2}\right]t_m$$

$$+\frac{2(\sqrt{l}-1)}{M}\sum_{m=1}^{M}q_{N+m}\left[\frac{1}{z_k^2-t_m^2} + \nu\frac{z_k^2-t_m^2}{(z_k^2-t_m^2)^2+4h^2}\right]t_m - \pi\varkappa = 0 \quad (k = 1, 2, \ldots, N)$$

$$-\frac{1}{N}\sum_{j=1}^{N}\frac{2ht_j\gamma_j}{(z_k^2-t_j^2)^2+4h^2} + \frac{2}{N}\sum_{m=1}^{N}q_m\left[\frac{1}{z_k^2-t_m^2} + \nu\frac{z_k^2-t_m^2}{(z_k^2-t_m^2)^2+4h^2}\right]t_m$$

$$+\frac{2(\sqrt{l}-1)}{M}\sum_{m=1}^{M}q_{N+m}\left[\frac{1}{z_k^2-t_m^2} + \nu\frac{z_k^2-t_m^2}{(z_k^2-t_m^2)^2+4h^2}\right]t_m - \pi\varkappa = 0 \quad (k = 1, 2, \ldots, M)$$

$$\frac{1}{N}\sum_{m=1}^{N}t_m q_m + \frac{\sqrt{l}-1}{M}\sum_{m=1}^{M}t_m q_{N+m} = 0 \qquad (2.3)$$

for the system of integral equations (2.1).

Having solved system (2.3), we determine the hydrodynamic characteristics and the cavity thickness by the formulas

$$C_y = \frac{2}{N}\sum_{j=1}^{N} t_j\gamma_j, \quad C_m = \frac{2}{N}\sum_{j=1}^{N} t_j^3\gamma_j, \quad x_c = \frac{C_y}{C_m}$$

$$C_d = -\frac{2}{N}\sum_{j=1}^{N}\left(\frac{dy}{dx}\right)_{t_j} t_j\gamma_j$$

$$t_n = \frac{2}{N}\sum_{m=1}^{n} t_m q_m \quad (n \leqslant N) \tag{2.4}$$

$$t_n = \frac{2}{N}\sum_{m=1}^{N} t_m q_m + \frac{\sqrt{l}-1}{M}\sum_{m=1}^{n} t_m q_{m+N} \quad (n > N)$$

The mean line $f_c(x)$ is determined from the first of Eqs. (2.1).

The calculations for N = 8 and M = 16, and various lengths of cavern and various h were carried out on a BÉSM-2M computer. The results of calculations of $\varkappa$, γ, and C_y are shown in Figs. 1-3. In Fig. 4 is shown a comparison of the results obtained by the described linearized theory (solid lines) and those of the exact solution (dashed lines) derived by the nonlinear theory [8]. This shows that for a plate fully cavitating under a free surface the linear theory yields higher values of lift.

For $h \to \infty$ the correlation between the calculated results and those of the exact solution presented by Guerst [7] (dashed lines in Figs. 1 and 3) is good. The error of determination of the over-all characteristics does not exceed 4%, and decreases with increased cavity length.

Example 2. Pressure distribution is specified by a rectangular law, i.e., $\gamma(x) = A = \text{const}$.

The unknown q(x) and $\varkappa$ are determined from the last two equations of system (1.9), while its first equation is used for determining the mean line $f_c(x)$.

Using the method of discrete perturbations, we obtain for q_j and $\varkappa$ the system of linear algebraic equations

$$\frac{2}{N}\sum_{j=1}^{N} q_j\left[\frac{N}{(i-{}^3/_4)-(j-{}^1/_4)} + \nu\frac{(i-{}^3/_4)-(j-{}^1/_4)}{[(i-{}^3/_4)-(j-{}^1/_4)]^2+4h^2N}\right] + \frac{2(l-1)}{M}\sum_{j=1}^{M} q_{N+j}\left[\frac{MN}{(i-{}^3/_4)M-[1+(j-{}^1/_4)(l-1)N]}\right.$$

$$\left. + \nu\frac{(i-{}^3/_4)M-[1+(j-{}^1/_4)(l-1)N]}{\{(i-{}^3/_4)M-[1+(j-{}^1/_4)(l-1)N]\}^2+4h^2MN}\right] - 2\pi\varkappa$$

$$= A\left[\operatorname{arc\,tg}\frac{(i-{}^3/_4)}{2hN} - \operatorname{arc\,tg}\frac{(i-{}^3/_4)-N}{2hN} - \pi\right] \quad (i = 1, \ldots, N)$$

$$\frac{2}{N}\sum_{j=1}^{N} q_j\left[\frac{MN}{1+(i-{}^3/_4)(l-1)N-(j-{}^1/_4)M}\right.$$

$$\left. + \nu\frac{1+(i-{}^3/_4)(l-1)N-(j-{}^1/_4)M}{[1+(i-{}^3/_4)(l-1)N-(j-{}^1/_4)M]^2+4h^2MN}\right]$$

$$+\frac{2(l-1)}{M}\sum_{j=1}^{M} q_{N+j}\left[\frac{M}{1+(i-{}^3/_4)(l-1)-[1+(j-{}^1/_4)(l-1)]} + \nu\frac{1+(i-{}^3/_4)(l-1)-[1+(j-{}^1/_4)(l-1)]}{\{[1+(i-{}^3/_4)(l-1)]-[1+(j-{}^1/_4)(l-1)]\}^2+4h^2M}\right]$$

$$-2\pi\varkappa = A\left[\operatorname{arc\,tg}\frac{1+(i-{}^3/_4)(l-1)}{2hM} - \operatorname{arc\,tg}\frac{(i-{}^3/_4)(l-1)}{2hM}\right] \quad (i = 1, 2, \ldots, M)$$

$$\frac{1}{N}\sum_{j=1}^{N} q_j + \frac{l-1}{M}\sum_{j=1}^{M} q_{N+j} = 0 \tag{2.5}$$

The results of calculations are presented in Fig. 5.

The described numerical method can also be used for calculating flows past cavitating grids and for determining the wall effect in hydrodynamic tubes. It can also be extended to the calculation of slender supercavitating load-carrying surfaces.

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